International Journal of Theoretical Physics, Vol. 4, No. 1 (1971), pp. 19–32

Classical Spin Variables and Classical Counterpart of the Dirac–Feynman–Gell-Mann Equation

S. DEPAQUIT, PH. GUÉRET AND J. P. VIGIER

Institut Henri Poincaré, Paris

Received: 28 *June* 1970

Abstract

In an extended relativistic fluid droplet, it is possible to define new internal variables which correspond to the classical counterpart of spin. If we introduce a new constraint, different from Weysenhoff's, we obtain by quantisation the Feynman–Gell-Mann wave equation. This also yields a theoretical connection between mass and spin which can be compared with the observed baryon boson mass spectrum.

Since the very beginning of spin theory, most physicists have accepted, without criticism, the famous statement of Pauli, that the spin has no classical counterpart and must be considered as a purely quantum mechanical concept tied with quantum mechanical matrices, related with finite group representations. This is a very strong statement from the physical point of view. Its logical origin rests, of course, in the classical point particle picture which leaves no room for classical spin variables.

The aim of the present article is to show that Pauli was wrong on this point. If one starts from a relativistic model of rotating fluid masses, one can find a classical model of spin and show that the quantisation of a special solution of its internal motions leads to the Feynman–Gell-Mann wave equation, which is equivalent to Dirac's. Of course, such a model ties mass and spin together and we shall discuss this relation in connection with the baryon mass spectrum. The bosons correspond to a different internal rotational symmetry.

This result can be obtained if we combine the relativistic model of rotating fluid masses, elaborated by Bohm & Vigier (1958), with the mathematical analysis of spinor variables, made by Hara & Goto (1968), in order to study extended models of elementary particles.

As one known (Bohm & Vigier, 1958), if we introduce within the relativistic droplet a symmetry energy-momentum density $T_{\mu\nu}$ (with $\partial^\nu T_{\mu\nu} = 0$) and current density j_μ (with $\partial^\mu j_\mu = 0$), we can define:

(*a*) a total momentum $G_\mu = \int T_{\mu 0}\, dV =$ constant (with $dG_\mu/dt = 0$), where dV represents the element of volume in any frame Σ. From this definition, one can take a special rest frame Π_0 in which $G_l = 0$.

(*b*) in Π_0, a centre-of-matter density (c.m.d.) by

$$Y_i^0 J_0^0 = \int_{\Pi_0} j_0^0 x_i^0 \, dV^0 \tag{1}$$

(the superscript zero denoting all quantities which refer to the frame Π_0), $J_0^0 = \int j_0^0 dV^0$, and a four-velocity $v_\mu = \dot{Y}_\mu = (d/d\tau) Y_\mu$, τ representing the proper time along the world-line followed by the c.m.d. ($v_\mu v^\mu = -c^2$).

(*c*) in the rest frame Σ_0 of the c.m.d. ($v_i = 0$), a centre of mass (c.m.) by

$$G_0 X_i = \int_{\Sigma_0} T_{00} x_i \, dV \tag{2}$$

The 4-velocity u_μ of the c.m. is proportional to G_μ, since we have (Bohm & Vigier, 1958)

$$u_\mu = \frac{G_\mu}{M_0 c} \tag{3}$$

with

$$M_0^2 c^2 = -G_\mu G^\mu \tag{4}$$

Let us introduce the inner angular momentum of the fluid droplet with regard to the c.m.d. as

$$M_{\mu\nu} = \int [(x_\mu - Y_\mu) T_{0\nu} - (x_\nu - Y_\nu) T_{\mu 0}], dV \tag{5}$$

we can express the total angular momentum $L_{\mu\nu}$ with regard to an arbitrary frame by

$$L_{\mu\nu} = M_{\mu\nu} + Y_\mu G_\nu - Y_\nu G_\mu \tag{6}$$

so that its conservation (we have always $\dot{L}_{\mu\nu} = 0$) yields the final set of relations

$$\dot{M}_{\mu\nu} = G_\mu \dot{Y}_\nu - G_\nu \dot{Y}_\mu, \ \dot{G}_\mu = 0 \tag{7}$$

which would be completed by three constraints to describe the total motion of the c.m.d.

Until now, in the literature (Halbwachs, 1960), the constraints have been Weysenhoff's: $M_{\alpha\beta} \dot{Y}^\beta = 0$, or some of its generalisations and one knows that their quantisation *does not yield the usual spin equations* (Corben, 1968).

In this work, we now take, instead of $M_{\alpha\beta} \dot{Y}^\beta = 0$, the new constraint relations

$$\dot{\omega}_{\alpha\beta} G^\beta = 0 \tag{8}$$

where $\omega_{\alpha\beta}$ represent the relativistic angular variables canonically conjugated to $M_{\alpha\beta}$ as the Y_μ are conjugated with the G_μ's: the $\dot{\omega}_{\alpha\beta}$, representing the average angular velocity of the droplet as whole around the c.m.d. Of course, this model is related to Yukawa's famous bilocal structure of elementary particles (Yukawa, 1953). The physical meaning of the new condition is clear: in Π_0 the particle undergoes a purely spatial rotation.

The next step is to quantise this model. Let us first recall a simple method of quantisation, in the simple case of a spinless point particle. Introducing its position x_μ as function of the proper time τ along the world-line followed, and the associated scalar Hamiltonian (which must be equal to $-mc^2$), namely

$$H = \left(\frac{G_\mu G^\mu}{2m} - \frac{mc^2}{2}\right) \tag{9}$$

we have $\dot{x}_\mu = \partial H/\partial G^\mu = G_\mu/m$ and $\dot{G}_\mu = -\partial H/\partial x^\mu = 0$ (de Broglie *et al.*, 1963). One quantises by introducing a scalar wave field $\Phi(x_\mu, \tau)$, writing $H = -i\hbar\partial/\partial\tau = -i\hbar\partial_\tau$, $G_\mu = -i\hbar\partial_\mu$. We get the generalised Schrödinger equation

$$-i\hbar\,\partial_\tau \Phi(x_\mu, \tau) = H\Phi(x_\mu, \tau) \tag{10}$$

Observed physical waves correspond to the stationary solution

$$\Phi(x_\mu, \tau) = \exp\left(-\frac{imc^2\tau}{\hbar}\right)\Psi(x_\mu) \tag{11}$$

which satisfies the usual wave equation

$$\Box\Psi(x_\mu) = \frac{m^2c^2}{h}\Psi(x_\mu) \tag{12}$$

In our general case, we can take

$$\begin{aligned} H &= \left(\frac{1}{2m}G_\mu G^\mu + \frac{a}{2m^2c^2}G_\mu M^{\mu\nu}G^\sigma M_{\sigma\nu} + \frac{m_0{}^2c^2}{2}\right) \\ &= \left(\frac{1}{2m}G_\mu G^\mu + \frac{a}{2m^2c^2}S^\nu S_\nu + \frac{m_0{}^2c^2}{2}\right) \end{aligned} \tag{13}$$

where $S_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^\nu M^{\alpha\beta} = G^\nu \bar{M}_{\mu\nu}$ is the well-known definition of the spin in the Poincaré group, m, m_0 and a are constants. This is the most general Hamiltonian (up to higher order terms), invariant under the Poincaré group.

This yields immediately

$$\left|\begin{aligned} \dot{Y}_\mu &= \frac{\partial H}{\partial G^\mu} = \frac{G_\mu}{m} + \frac{a}{m^2c^2}G_\sigma \bar{M}^{\sigma\nu}M_{\mu\nu} = \frac{G_\mu}{m} + \frac{a}{m^2c^2}M_{\mu\nu}S^\nu \\ \dot{\omega}_{\mu\nu} &= \frac{\partial H}{\partial M^{\mu\nu}} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^\alpha S^\beta \end{aligned}\right. \tag{14}$$

with $\dot{G}_\mu = -\partial H/\partial Y^\mu = 0$, $\dot{M}_{\mu\nu} = -\partial H/\partial\omega^{\mu\nu} = 0$, $G^\mu\dot{\omega}_{\mu\nu} = 0$ and $\ddot{Y}_\mu = \dot{S}_\mu = 0$. Moreover, we have (Takabayasi, 1966): $G_\alpha G^\alpha M_{\mu\nu} = r[_\mu G_\nu] + i\epsilon_{\mu\nu\rho\sigma}G^\rho S^\sigma$: the total angular momentum is the sum of the orbital momentum of the c.m. with the spin. We have also $\dot{Y}_\mu = \dot{r}_\mu = d/d\tau(M_{\mu\nu}G^\nu)$.

We shall now show (in the case of spin $\frac{1}{2}$; the reasoning can be developed identically for any spin) that the corresponding quantisation yields the

Dirac–Feynman–Gell-Mann equations. Indeed, let us work in the Π_0 frame ($G_i = 0$) centred on the c.m., we get

$$H = \left(\frac{1}{2m} G^4 G_4 + \frac{a}{2m^2 c^2} G_4 M^{ij} G^4 M_{ij} + \frac{m_0{}^2 c^2}{2}\right) \tag{15}$$

The quantisation could be obtained directly by the substitution $G_4 \to -i\hbar\partial/\partial t$. $M_{ij} \to -i\hbar\partial/\partial\omega^{ij}$ and proceeding as before. However, in order to clarify the physical meaning of spinors, it is preferable to change variables and define the angular velocity with the help of the rotation of a tetrad $b_\mu{}^\xi$ tied to the c.m.d. (the body frame) with regard to a tetrad $a_\mu{}^\xi$ tied to the external observer (the observer frame).

We define a general frame in Minkowski space by four, orthogonal, unitary vectors $a_\mu{}^\xi$ or $b_\mu{}^\xi$ ($\xi = 1, 2, 3, 4$) satisfying the orthonormality conditions

$$a_\mu{}^\xi a_\nu{}^\xi = \delta_{\mu\nu}, \qquad a_\mu{}^\xi a_\mu{}^\eta = \delta^{\xi\eta}, \qquad b_\mu{}^\xi b_\nu{}^\xi = \delta_{\mu\nu}, \qquad b_\mu{}^\xi b_\mu{}^\eta = \delta^{\xi\eta} \tag{16}$$

The time-like vectors being ia_μ and ib_μ. (We have $x_4 = ix_0 = ict$) and the angular velocity can be written $\dot\omega_{\mu\nu} = b_\mu{}^\xi b_\nu{}^\xi$. If we start from a fixed reference frame $a_\mu{}^\xi$ the transition from this frame to any orientation of the moving frame $b_\mu{}^\xi$ is given by the expression

$$\begin{vmatrix} b_\mu{}^1 \\ b_\mu{}^2 \\ b_\mu{}^3 \\ b_\mu{}^4 \end{vmatrix} = \begin{vmatrix} \cos\phi^+/2 & \sin\phi^+/2 & 0 & 0 \\ -\sin\phi^+/2 & \cos\phi^+/2 & 0 & 0 \\ 0 & 0 & \cos\phi^+/2 & \sin\phi^+/2 \\ 0 & 0 & -\sin\phi^+/2 & \cos\phi^+/2 \end{vmatrix} \times$$

$$\times \begin{vmatrix} \cos\phi^-/2 & \sin\phi^-/2 & 0 & 0 \\ -\sin\phi^-/2 & \cos\phi^-/2 & 0 & 0 \\ 0 & 0 & \cos\phi^-/2 & -\sin\phi^-/2 \\ 0 & 0 & \sin\phi^-/2 & \cos\phi^-/2 \end{vmatrix} \times$$

$$\times \begin{vmatrix} \cos\theta^+/2 & 0 & -\sin\theta^+/2 & 0 \\ 0 & \cos\theta^+/2 & 0 & -\sin\theta^+/2 \\ \sin\theta^+/2 & 0 & \cos\theta^+/2 & 0 \\ 0 & \sin\theta^+/2 & 0 & \cos\theta^+/2 \end{vmatrix} \times$$

$$\times \begin{vmatrix} \cos\theta^-/2 & 0 & -\sin\theta^-/2 & 0 \\ 0 & \cos\theta^-/2 & 0 & \sin\theta^-/2 \\ \sin\theta^-/2 & 0 & \cos\theta^-/2 & 0 \\ 0 & -\sin\theta^-/2 & 0 & \cos\theta^-/2 \end{vmatrix} \times$$

$$\times \begin{vmatrix} \cos\psi^+/2 & \sin\psi^+/2 & 0 & 0 \\ -\sin\psi^+/2 & \cos\psi^+/2 & 0 & 0 \\ 0 & 0 & \cos\psi^+/2 & \sin\psi^+/2 \\ 0 & 0 & -\sin\psi^+/2 & \cos\psi^+/2 \end{vmatrix} \times$$

$$\times \begin{vmatrix} \cos\psi^-/2 & \sin\psi^-/2 & 0 & 0 \\ -\sin\psi^-/2 & \cos\psi^-/2 & 0 & 0 \\ 0 & 0 & \cos\psi^-/2 & -\sin\psi^-/2 \\ 0 & 0 & \sin\psi^-/2 & \cos\psi^-/2 \end{vmatrix} \begin{vmatrix} a_\mu{}^1 \\ a_\mu{}^2 \\ a_\mu{}^3 \\ a_\mu{}^4 \end{vmatrix} \quad (17)$$

where the angles are defined by

$$\begin{aligned} \phi^+ &= \phi_1 + i\phi_2, & \phi^- &= \phi_1 - i\phi_2 \\ \theta^+ &= \theta_1 + i\theta_2, & \theta^- &= \theta_1 - i\theta_2 \\ \psi^+ &= \psi_1 + i\psi_2, & \psi^- &= \psi_1 - i\psi_2 \end{aligned} \quad (18)$$

ϕ_1, θ_1, ψ_1 being the ordinary space Euler angles and $i\phi_2$, $i\theta_2$, $i\psi_2$ hyperbolic angles describing pure Lorentz transforms (Hillion & Vigier, 1960).†

It can easily be shown that a set of self-dual bivectors built with the help of the $b_\mu{}^\xi$

$$B^{r\pm} = b_k{}^r b_4{}^4 - b_4{}^r b_k{}^4 \pm \epsilon_{ijk} b_i{}^r b_j{}^4, \qquad i, j, k, r = 1, 2, 3 \quad (19)$$

can be obtained from the corresponding expressions in $a_\mu{}^\xi$

$$A_k^{r\pm} = a_k{}^r a_4{}^4 - a_4{}^r a_k{}^4 \pm \epsilon_{ijk} a_i{}^r a_j{}^4 \quad (20)$$

by the transformations

$$\begin{vmatrix} B^{1+} \\ B^{2+} \\ B^{3+} \end{vmatrix} = \begin{vmatrix} \cos\phi^+\cos\theta^+\cos\psi^+ - \sin\phi^+\sin\psi^+ & \cos\phi^+\cos\theta^+\sin\psi^+ + \sin\phi^+\cos\psi^+ & -\cos\phi^+\sin\theta^+ \\ -\sin\phi^+\cos\theta^+\cos\psi^+ - \cos\phi^+\sin\psi^+ & -\sin\phi^+\cos\theta^+\sin\psi^+ + \cos\phi^+\cos\psi^+ & -\sin\phi^+\sin\theta^+ \\ \sin\theta^+\cos\psi^+ & \sin\theta^+\sin\psi^+ & \cos\theta^+ \end{vmatrix} \begin{vmatrix} A^{1+} \\ A^{2+} \\ A^{3+} \end{vmatrix} \quad (21)$$

that is exactly the usual non-relativistic expression in Euler angles, where complex Euler angles $\omega^\pm = (\phi^\pm, \theta^\pm, \psi^\pm)$ have replaced real ones, the transition from A_k^{r+} to B_k^{r+} is determined by the ω^+ only, the A_k^{r-}, B_k^{r-} by the ω^- only. This means, as Einstein and Mayer have noticed (1932), that we can represent any Lorentz transform by two complex–conjugate three-dimensional rotations. The explicit form of these representations can be obtained by writing the infinitesimal rotation operators corresponding to these complex rotations.

A short calculation (Halbwachs *et al.*, 1959) gives for rotations around the bivectors $A_k^{r\pm}$

$$\left| \begin{aligned} J_1{}^\pm &= -\sin\phi^\pm p_{\theta\pm} - \cot g\theta^\pm \cos\phi^\pm p_{\phi\pm} + \frac{\sin\theta^\pm}{\cos\phi^\pm} p_{\psi\pm} \\ J_2{}^\pm &= \cos\phi^\pm p_{\theta\pm} - \cot g\theta^\pm \sin\phi^\pm p_{\phi\pm} + \frac{\sin\phi^\pm}{\sin\theta^\pm} p_{\psi\pm} \\ J_3{}^\pm &= p_{\phi\pm} \end{aligned} \right.$$

† For clarity, we have explicitly reproduced this connection between the spinors and relativistic Euler angles of the paper.

for rotations around the bivectors $B_k^{r\pm}$

$$\left| \begin{aligned} J'^{\pm}_1 &= \sin\psi^{\pm} p_{\theta^{\pm}} + \cot g\theta^{\pm} \cos\psi^{\pm} p_{\psi^{\pm}} - \frac{\cos\psi^{\pm}}{\sin\theta^{\pm}} p_{\phi^{\pm}} \\ J'^{\pm}_2 &= \cos\psi^{\pm} p_{\theta^{\pm}} - \cot g\theta^{\pm} \sin\psi^{\pm} p_{\psi^{\pm}} + \frac{\sin\psi^{\pm}}{\sin\theta^{\pm}} p_{\phi^{\pm}} \\ J'^{\pm}_3 &= p_{\psi^{\pm}} \end{aligned} \right.$$

here

$$p_{\theta^{\pm}} = -j\hbar\frac{\partial}{\partial\theta^{\pm}}, \qquad p_{\phi^{\pm}} = -j\hbar\frac{\partial}{\partial\phi^{\pm}}, \qquad p_{\psi^{\pm}} = -j\hbar\frac{\partial}{\partial\psi^{\pm}} \qquad (j = \sqrt{-1})$$

The difference between the two symbols i and j, both with squares equal to -1, has been introduced by Möller (1949), Synge (1954), and others in order to avoid confusion. It can be immediately noticed that the preceding operations which are not hermitian in the common sense of the word (Hillion & Vigier, 1959) satisfy the commutation relations of the three-dimensional rotation group

$$(J_k^{\pm}, J_j^{\pm}) = -jJ_i^{\pm}, \qquad (J'^{\pm}_k, J'^{\pm}_j) = -jJ'^{\pm}_i$$

(i, j, k, is a circular permutation of 1, 2, 3).

We also get the relations

$$(J_k^{+}, J_j^{-}) = 0, \qquad (J_k^{+}, J'^{-}_j) = 0, \qquad (J'^{+}_k, J'^{-}_j) = 0$$

which show that the J^+ and J^- are independent.

Introducing further the two operators

$$(J^+)^2 = (J'^+)^2 \qquad \text{and} \qquad (J^-)^2 = (J'^-)^2$$

we obtain

$$(J_3^+, (J^+)^2) = 0, \qquad (J_3^-, (J^-)^2) = 0, \qquad (J'^+_3 (J'^+)^2) = 0$$
$$(J'^-_3, (J'^-)^2) = 0, \qquad ((J^+)^2, (J^-)^2) = 0$$

It is now clear that we can satisfy a Lorentz transform by the simultaneous eigenfunctions of three commuting operators among the preceding ones, namely

$$J_3^+, J'^+_3, (J^+)^2$$

The explicit form for these eigenfunctions is given by expression

$$\begin{aligned} Y_j^{m^+, m'^+}(\omega^+) &= \left(\frac{\sin\theta^+}{2}\right)^{-m^+ + m'^+} \left(\frac{\cos\theta^+}{2}\right)^{-m^+ - m'^+} \exp\left[j(m^+\phi^+ + m'\psi^+)\right] \times \\ &\times \frac{d^{j^+ - m^+}}{d(\sin^2\theta^+/2)^{j^+ - m'^+}} \left[\left(\frac{\sin\theta^+}{2}\right)^{2j^+ - 2m'^+} \left(\frac{\cos\theta^+}{2}\right)^{2j^+ + 2m^+}\right] \\ &= \theta_{j^+}^{m^+, m'^+}(\theta^+) \exp\left[j(m^+\phi^+ + m'^+\psi^+)\right] \end{aligned}$$

and we have shown in another paper (Hillion & Vigier, 1969) that the number j^+ may take integer or half integer values, the corresponding values of m^+ and m'^+ being

$$-j^+, -j^+ - 1, \ldots, j^+ - 1, j^+$$

Obviously, the same results apply to angles ω^-.

The products of functions $Y_{j^+}^{m^+, m'^+}(\omega^+)\, Y_{j^-}^{m^-, m'^-}(\omega^-)$ transform under the $D(j^+, j^-)$ representation of the proper Lorentz group. More precisely we can build functions which transform under the representation

$$D(j^+, j^-) \oplus D(j^-, j^+)$$

of the full Lorentz group. To do this we use the eigenfunctions of the six commuting operators

$$(J^+)^2, J_3^+, (J^-)^2, J_3^-, S'^2, S_3' \qquad (\text{with } S_k' = J_k'^+ + J_k'^-,\ S'^2 = S_k' S_k')$$

These eigenfunctions are series of linear products of the

$$Y_{j^+}^{m^+, m'^+}(\omega^+)\, Y_{j^-}^{m^-, m'^-}(\omega^-)$$

eigenfunctions multiplied by suitable Clebsch–Gordan coefficients

$$Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = \sum_{-m'^+, -m'^-} (j^+, j^-, -m'^+, -m'^- \mid j^+, j^-, s', -m')\, Y_{j^+}^{m^+, m'^+}(\omega^+)\, Y_{j^-}^{m^-, m'^-}(\omega^-) \tag{22}$$

$$s' = j^+ + j^-, j^+ + j^- - 1, \ldots |j^+ - j^-|$$

$$m' = -s', -s' + 1, \ldots, s' - 1, s'$$

and we have

$$(J^\pm)^2 Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = j^\pm(j^\pm + 1) Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-)$$

$$J_3^\pm Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = m^\pm Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-)$$

$$S'^2 Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = s'(s' + 1) Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-)$$

$$S_3' Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = m' Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-)$$

Moreover (Dragt, 1965)

$$PZ_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = (-1)^{j^+ + j^- - s'} Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-)$$

$$CZ_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = (-1)^{|m^+ + m^- - m'|} Z_{j^+, j^-, s'}^{-m^+, -m^-, -m'}(\omega^+, \omega^-) \tag{23}$$

P and C being, respectively, the parity and charge conjugation operators.

Now we can establish (Hillion & Vigier, 1959) the following theorem: If a set of functions $Z_{j^+, j^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-)$, we fix the values of j^+, j^-, s', m', the corresponding set transforms like the representation

$$D(j^+, j^-) \oplus D(j^-, j^+)$$

of the full Lorentz group. In other words, the functions of this set constitute the basic frame of a finite-dimensional vector space transforming into itself under the Lorentz group according to the corresponding representation.

These spaces are, of course, subspaces of the general enumerably infinite dimensional Hilbert space containing all finite-dimensional representations of the full Lorentz group.

For example, for spin $\frac{1}{2}$, we have in the right-handed frame $j^+ = 0$, $j^- = \frac{1}{2}$, $s' = \frac{1}{2}$, two Feynman–Gell-Mann two components spinors

$$\xi = \phi^s = \begin{vmatrix} Z^{0,\,1/2,\,1/2}_{0,\,1/2,\,1/2}(\omega^+, \omega^-) \\ Z^{0,\,-1/2,\,1/2}_{0,\,1/2,\,1/2}(\omega^+, \omega^-) \end{vmatrix} = \begin{vmatrix} Y^{1/2,\,1/2}_{1/2}(\omega^-) \\ Y^{-1/2,\,1/2}_{1/2}(\omega^-) \end{vmatrix}$$

$$\eta = \phi_r = \begin{vmatrix} -Z^{0,\,-1/2,\,1/2}_{0,\,1/2,\,1/2}(\omega^+, \omega^-) \\ Z^{0,\,1/2,\,1/2}_{0,\,1/2}(\omega^+, \omega^-) \end{vmatrix} = \begin{vmatrix} -Y^{-1/2,\,1/2}_{1/2}(\omega^-) \\ Y^{1/2,\,1/2}_{1/2}(\omega^-) \end{vmatrix} \tag{24}$$

and similar expressions for $\phi_{\dot{r}}$, $\phi^{\dot{s}}$ and $m' = -\frac{1}{2}$.

The quantisation is now straightforward. Clearly, the ω_{ij} of the hydrodynamic model corresponds to the projections on the body frame; that is, to the S_k' operators. $J(J+1)$ are eigenvalues of $S'^k S_k'$.

We thus write

$$H = \left(-\frac{1}{2m} \partial^t \partial_t{}^t - \frac{1}{2m^2 c^2} \partial^t \partial_t S'^k S_k' + \frac{m_0 c^2}{2} \right) \tag{25}$$

in which $(1/2m^2 c^2) S'^k S_k'$ can take two typical forms,† namely (Hara & Goto, 1968)

$$-\mathscr{H}_s = \frac{1}{2I_1} S'^2$$

if the body is spherical,

$$-\mathscr{H}_c = \frac{1}{2I_1} S'^2 + \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right)(S_3')^2$$

if the body has cylindrical symmetry.

We quantise, as before, by introducing a total scalar field

$$\Phi\,(\tau, Y_\mu, \omega^+, \omega^-) = \exp\left(-\frac{imc^2\,\tau}{\hbar}\right) \sum \Psi^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(Y_\mu) . Z^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(\omega^+, \omega^-) \tag{26}$$

with

$$\Psi^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(Y_\mu) = \int \Phi\ . Z^{*0,\,m^-,\,m'}_{\ \,0,\,j^-,\,s'}(\omega^+, \omega^-)\, d\omega \tag{27}$$

where $d\omega$ is the volume element of the real part of the Euler angles. This implies that the $\Psi^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(Y_\mu)$ form a two-components spinor transforming contragrediently to the $Z^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(\omega^+, \omega^-)$. Introducing (26) into (10), we see that they satisfy the Feynman–Gell-Mann equation:

$$\Box \Psi^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(Y_\mu) = \frac{m^2 c^2}{\hbar} \Psi^{0,\,m^-,\,m'}_{0,\,j^-,\,s'}(Y_\mu)$$

† Inside the particle, space-time curvature can rise enormously so that $g_{ik} \gg 1$. Thus the $S_k' S'^k = S_k' g^{ik} S_i'$ terms are developed locally on diagonal covariant operators so that in general $S_k' S'^k = a S_k' S_k' + b(S_3')^2$, with a and b constants.

TABLE 1. $m_J^2 = m_0^2 + aJ$

	$\Lambda^{(P)}$ \ J	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{15}{2}$	$\frac{19}{2}$
Octets	$N^{(+)}$	939		1688					
			$a = 0{\cdot}98$						
	$\Lambda^{(+)}$	1115		1815		2350(?)			
			$a = 1{\cdot}03$						
				$a = 1{\cdot}07$					
	$\Sigma^{(+)}$	1192		1910		2455(?)			
			$a = 1{\cdot}11$						
				$a = 1{\cdot}15$					
	$\Xi^{(+)}$	1317		2090(?)		2500(?)			
			$a = 1{\cdot}19$						
				$a = 1{\cdot}13$					
Decuplets	$\Delta^{(+)}$		1236		1950		2420		
				$a = 1{\cdot}14$					
					$a = 1{\cdot}08$				
	$\Sigma^{(+)}$		1385		2030		2595(?)	2850(?)	3230(?)
				$a = 1{\cdot}10$					
					$a = 1{\cdot}20$				
						$a = 1{\cdot}10$			
							$a = 1{\cdot}11$		
	$N^{(-)}$		1520		2190				
				$a = 1{\cdot}24$					
	$\Lambda^{(-)}$		1520		2100				
				$a = 1{\cdot}05$					
	$\Sigma^{(-)}$		1670		2250(?)				
				$a = 1{\cdot}14$					
	$\Lambda^{(+)}$		1330(?)		2015				
				$a = 1{\cdot}14$					

Note. The symbol (?) denotes existing particles with undetermined spin values.

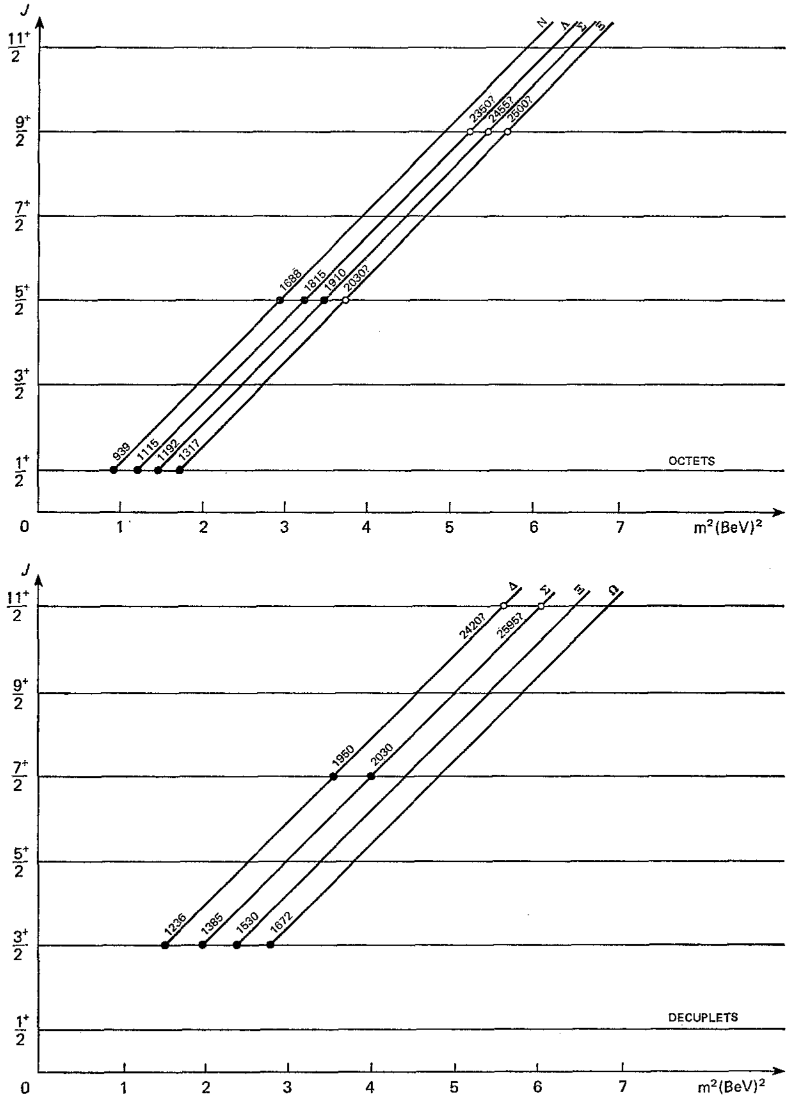

Figure 1.

with

(1) $m^2 = m_0{}^2 + aJ(J+1)$ (m_0 and a are constants), for spherical symmetry

(1) $m^2 = m_0{}^2 + \dfrac{1}{2I_1} J(J+1) + \dfrac{1}{2}\left(\dfrac{1}{I_3} - \dfrac{1}{I_1}\right) m'^2$, for cylindrical symmetry

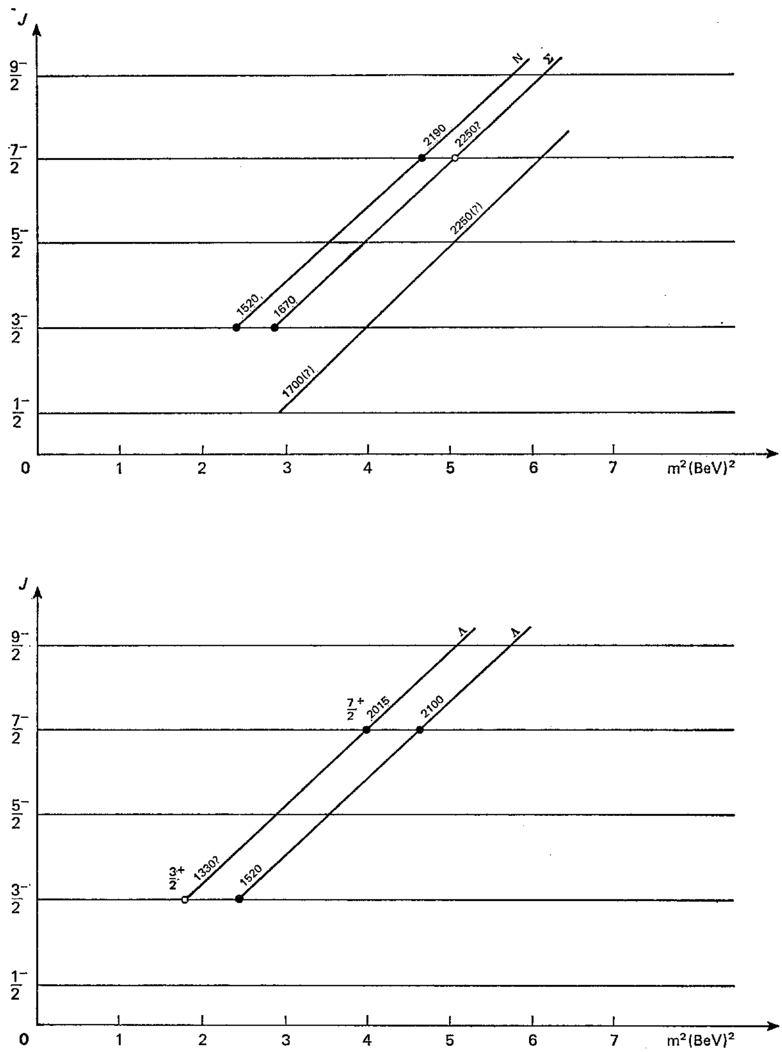

Figure 2.

This generalisation of the Dirac equation can be extended for any even and odd spin.

From the physical point of view, we shall discuss these two possibilities in connection with the observed baryon and boson mass spectra.

(I) If one considers baryons as compound quasi free 3-quarks states, we know (Dragt, 1965) that in the non-relativistic limit, they move, in the rest frame, in a two-dimensional space like plane so that we can consider their compound droplet structure to have cylindrical symmetry. If we further

TABLE 2. $m_J{}^2 = m_0{}^2 + aJ(J+1)$

I \ J	0	1	2	3
1	139	765	1320	1880
	$a = 0{\cdot}28$			
	$a = 0{\cdot}29$			
0	549	970(?)	1410	
	$a = 0{\cdot}28$			
0	720	1080	1514	
	$a = 0{\cdot}32$			
	$a = 0{\cdot}30$			
0	958	?	1650(?)	
	$a = 0{\cdot}29$			
1	1016	1235	1640(?)	2100(?)
	$a = 0{\cdot}25$			
	$a = 0{\cdot}28$			
0	1060	1285(?)	1660(?)	
	$a = 0{\cdot}27$			
	$a = 0{\cdot}27$			
0	1420			2380
$\frac{1}{2}$	494	892	1420	
	$a = 0{\cdot}28$			
	$a = 0{\cdot}29$			
$\frac{1}{2}$	725(?)	1080(?)		
	$a = 0{\cdot}32$			

Note. The symbol (?) denotes existing particles with undetermined spin values.

assume $I_1 \ll I_3$ (flat disk) and $m' = s' = J$, we get, for associated particles for which T, T_3 and Y are equal, the mass formula

$$m^2 = m_0{}^2 + aJ \qquad (a \text{ and } m_0 = \text{constants}) \tag{28}$$

Of course, it is a Regge-like formula obtained in a different way. It can be compared with a baryon resonances table. Indeed, if we construct a table

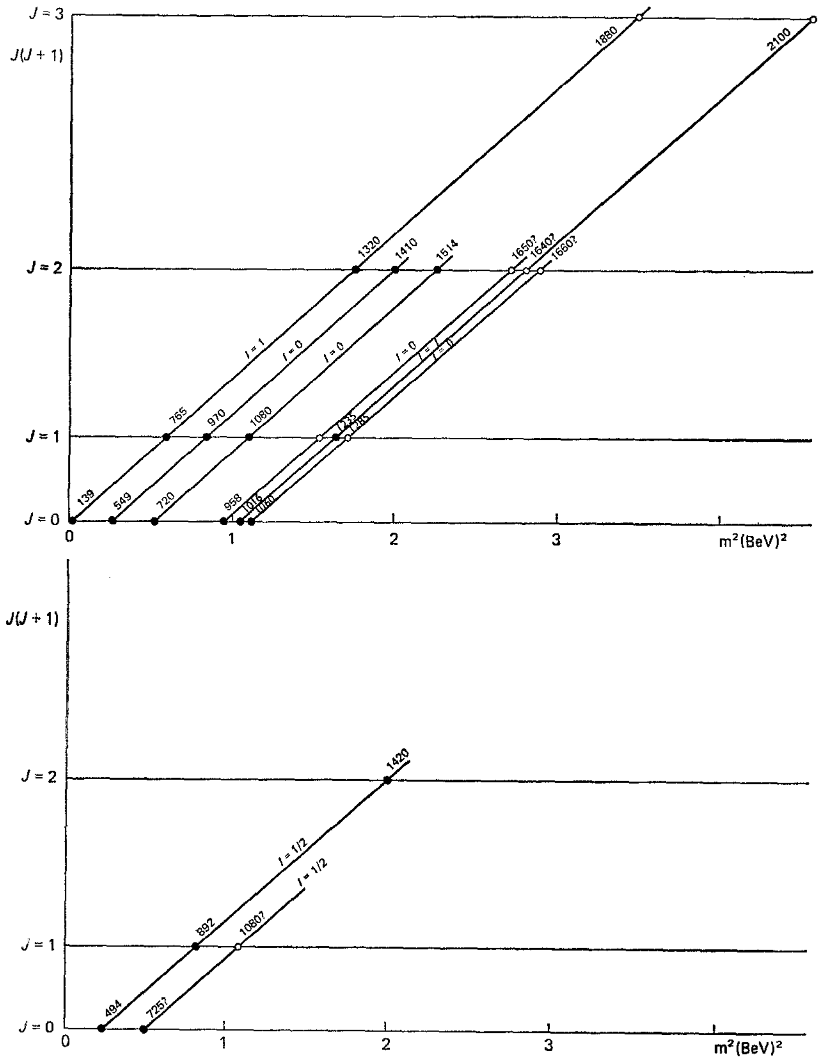

Figure 3.

where columns denote spin and lines correspond to associated experimental values of mass square, we get Table 1 which shows that a varies within very narrow limits. This can be further illustrated, if we plot as usual, m^2 versus J (Figs. 1 and 2), by quasi parallel lines connecting associated particles. Of course, a few resonant states remain outside and they could be considered as strongly bound baryon–boson states. Small deviations from ideal trajectories could be also explained by self energy contributions.

(II) If we further consider bosons as baryon–antibaryon compound states, we can assume that a 6-quarks-antiquarks state corresponds to a compound spherically symmetric droplet structure, so that we get, for mass formula of associated particles, the following expression

$$m_J^2 = m_0^2 + aJ(J+1) \tag{29}$$

This also fits astonishingly well with observed resonance data. Indeed, we get in this case Table 2, corresponding to Fig. 3.

If we further assume (Depaquit & Vigier, 1969) that bosons correspond to massive quanta emitted in baryon–baryon quantum jumps, we see the coefficient m_0 and a in (29) results from the m_0 and a values in (28). In our opinion, the theoretical advantage of this model is that is explains *both* baryon and boson mass spectrum J dependence in a simple way. It also paves the way for a physical description of the supplementary quantum numbers (T, T_3, Y, etc.) in terms of supplementary internal fluid excitations which imply as we shall show in a subsequent publication (Guéret & Vigier, 1971), a simple modification of (28) and (29), of the form

$$m^2 = m_0^2(1 + \Delta H) + aJ, \qquad m^2 = m_0^2(1 + \Delta H) + aJ(J+1) \tag{30}$$

ΔH describing a T, T_3, Y dependent perturbations.

Acknowledgements

The authors want to thank Professors L. de Broglie, H. Yukawa and D. Bohm for helpful suggestions and discussions.

References

Bohm, D. and Vigier, J. P. (1958). *Physical Review*, **109**, 6, 1882–1891.

de Broglie, L., Bohm, D., Hillion, P., Halbwachs, F., Takabayasi, T. and Vigier, J. P. (1963). *Physical Review*, **129**, 1, 438.

Corben, H. C. (1968). *Classical and Quantum Theories of Spinning Particles*. Holden-Day, San Francisco.

Depaquit, S. and Vigier, J. P. (1969). *Compte rendu hebdomadaire des séances de l'Académie Paris*, **268**, B, 657.

Dragt, A. J. (1965). *Journal of Mathematical Physics*, **6**, 4, 533.

Einstein, A. and Mayer, M. (1932). *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, 522.

Guéret, P. and Vigier, J. P., to be published.

Halbwachs, F. (1960). *Théories Relativistes des Fluides à Spin*. Gauthier-Villars, Paris.

Halbwachs, F., Hillion, P. and Vigier, J. P. (1959). *Annales de l'Institut Henri Poincaré*, **16**, 115.

Hara, O. and Goto, T. (1968). Preprint of the Department of Physics, NUP-A-68-8, Nihon University, Tokyo.

Hillion, P. and Vigier, J. P. (1959). *Annales de l'Institut Henri Poincaré*, **16**, 161.

Hillion, P. and Vigier J. P. (1960). *Nuclear Physics*, **16**, 2, 360.

Möller, C. (1949). *Annales de l'Institut Henri Poincaré*, **14**, 251.

Synge, J. L. (1954). *Relativity: The Special Theory*. North Holland Publ. Co.

Takabayasi, T. (1966). *Progress of Theoretical Physics*, **36**, 5, 1074.

Yukawa, H. (1953). *Physical Review*, **91**, 415.